

BPHO ROUND 1 SECTION 1 2025. AP SOLUTIONS.

- a) An Einstein is a unit corresponding to a mole of photons. How long does it take for a 630 nm and 0.50 mW laser to release an Einstein?

λ

P

[3]

* 1 mole is Avogadro's number of photons, $N_A = 6.02 \times 10^{23}$.

* Energy of a photon is $E = \frac{hc}{\lambda}$.

So if Pt is the energy released by the laser of power P in t seconds, the laser releases an Einstein in t seconds where

$$Pt = N_A \frac{hc}{\lambda} \Rightarrow \boxed{t = \frac{N_A hc}{\lambda P}}$$

$$\therefore t = \frac{6.02 \times 10^{23} \times 6.63 \times 10^{-34} \times 3.00 \times 10^8}{630 \times 10^{-9} \times 0.5 \times 10^{-3}} \text{ s}$$

$$= \boxed{3.8 \times 10^8 \text{ s}}$$

$$\approx 12 \text{ years.}$$

$$[1 \text{ year} = 365.25 \times 24 \times 3600 \text{ s}].$$

b) A loop of rope is wrapped around a tree trunk of negligible friction and clamped at A and pulled with a force F at D, as shown in Fig. 1.

- For what angle θ is the tension in the rope at AB, AC equal to the tension in the rope AD.
- For this condition, what is the length ℓ of the rope ABCA in terms of the radius of the tree r ? Express your answer in terms of fractions and surds (roots).

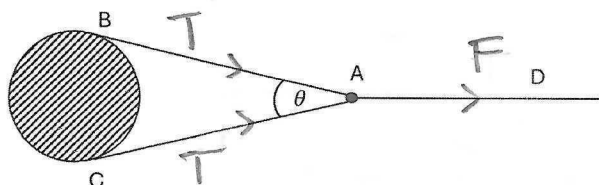
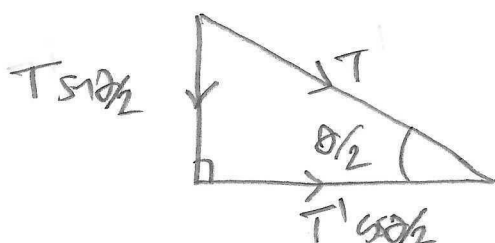


Figure 1: A tree trunk with a rope wrapped around and pulled by a force F .

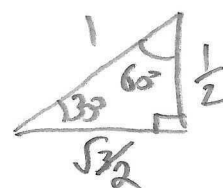
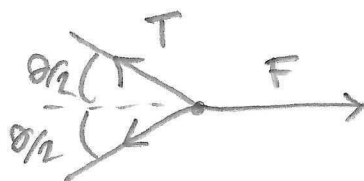
(i)



Assume $2T \cos \frac{\theta}{2} = F$

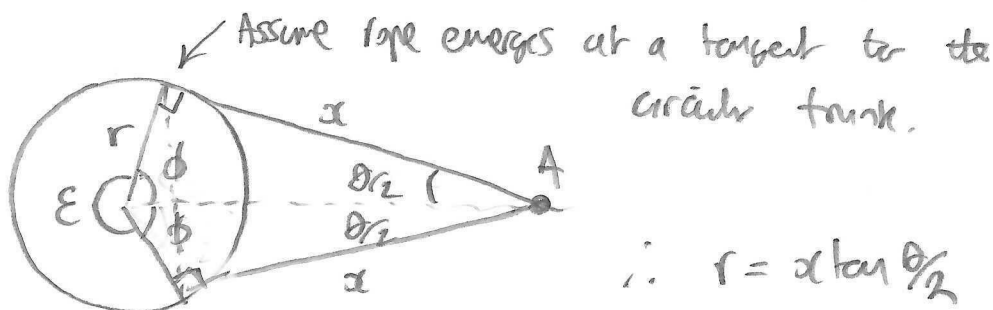
[4]

Since forces on clamp at A are



If $T = F$ this means $\cos \frac{\theta}{2} = \frac{1}{2} \Rightarrow \frac{\theta}{2} = 60^\circ$
 $\Rightarrow \theta = 120^\circ$

(ii)

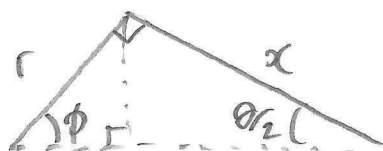


Assume rope enters at a tangent to the circular trunk.

$\uparrow$
 $\frac{2\pi}{3}$ radians

$\therefore r = x \tan \frac{\theta}{2}$

$\Rightarrow x = \frac{r}{\tan \frac{\theta}{2}}$



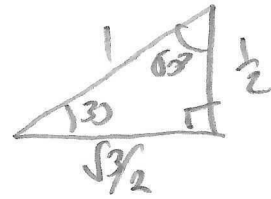
$\phi = 90^\circ - \frac{\theta}{2}$

Since $180^\circ = 90^\circ + \phi + \frac{\theta}{2}$

$\epsilon = 360^\circ - 2\phi = 360^\circ - 180^\circ + \theta \Rightarrow \epsilon = \frac{5\pi}{3} \text{ rad.}$
 $= \pi + \frac{2\pi}{3} \text{ rad}$

$$\begin{aligned} \text{So } l &= r\theta + 2x \\ &= \frac{\sqrt{11}}{3}r + \frac{2r}{\tan 60^\circ} \end{aligned}$$

$$= \frac{\sqrt{11}}{3}r + \frac{2r}{\sqrt{3}}$$



$$\therefore \boxed{l = r \left(\frac{\sqrt{11}}{3} + \frac{2}{\sqrt{3}} \right)} \quad \approx 6.39r.$$

- c) We can develop a simple model for traffic flow as follows; each car has a length L and travels at speed v , with the distance between the fronts of two consecutive cars being d , as shown in Fig. 2.

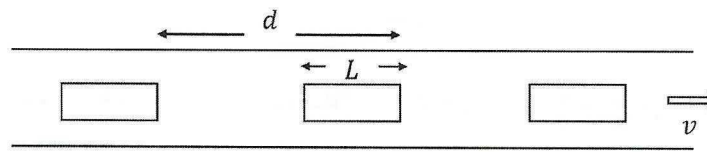


Figure 2: Flow of cars each of length L separated by distance d between car fronts.

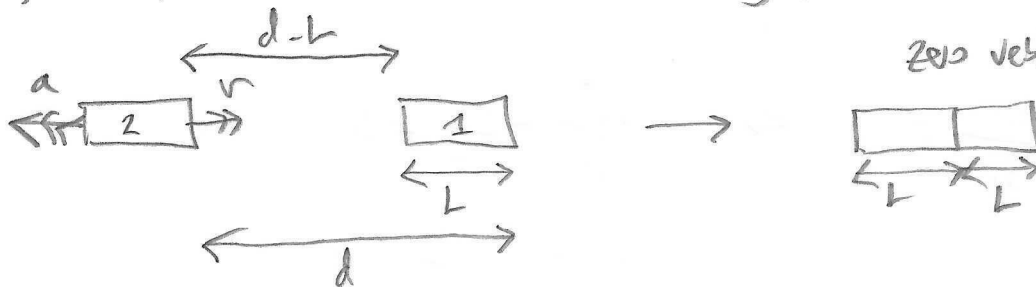
- Give the flow rate n of cars passing a point per unit time in terms of v and d .
Cars brake with a deceleration a and the drivers choose the value of d such that if the car in front were to stop instantaneously, the car behind could brake before hitting the back of the car in front.
- Obtain d in terms of v , a and L . Ignore the "thinking time".
- Find an expression for the optimum speed v for maximum flow rate in terms of L and a .

[4]

- (i) In t seconds, traffic flows a distance vt (at constant speed v). $\therefore$ # cars / second n is

$$n = \frac{vt}{d} = \boxed{\frac{v}{d}}$$

- (ii) If car 1 stops instantaneously, and car 2 brakes with deceleration a to zero velocity:



$$\therefore 2a(d-L) = v^2 \quad \left["v^2 = u^2 + 2ax" \text{ with } \begin{array}{l} v=0 \\ u=v \\ a=-a \\ x=d-L \end{array} \right]$$

OR, If cars have mass m

$$\underbrace{\frac{1}{2}mv^2}_{\text{KE of car 2}} = \underbrace{ma(d-L)}_{\text{work done by brakes.}}$$

$$\therefore \boxed{d = \frac{v^2}{2a} + L}$$

[If 'thinking time' Δt , reduce d by $v\Delta t$]

(iii) From (i): $n = \frac{v}{d} \Rightarrow d = \frac{v}{n}$

From (ii): $d = \frac{v^2}{2a} + L$

$\therefore$ Want $n(v)$ and desire max n . so find $\frac{dn}{dv} = 0$

So $\frac{v}{n} = \frac{v^2}{2a} + L$

$-\frac{1}{n^2} \frac{dn}{dv} v + \frac{1}{n} = \frac{v}{a}$
product rule

$$\left[\frac{d}{dv} (n^{-1}) = -n^{-2} \frac{dn}{dv} \right]$$

implicit differentiation.

So $\frac{dn}{dv} - \frac{n}{v} = -\frac{n^2 v}{a v}$

$$\frac{dn}{dv} = \frac{n}{v} - \frac{n^2}{a}$$

$\therefore \frac{dn}{dv} = 0 \Rightarrow \frac{n^2}{a} - \frac{n}{v} = 0$

$$n \left(\frac{n}{a} - \frac{1}{v} \right) = 0$$

Assuming $n > 0$

(it won't max n)

$\Rightarrow \frac{n}{a} = \frac{1}{v} \Rightarrow \boxed{n = \frac{a}{v}}$

when $n = \frac{a}{v}$,

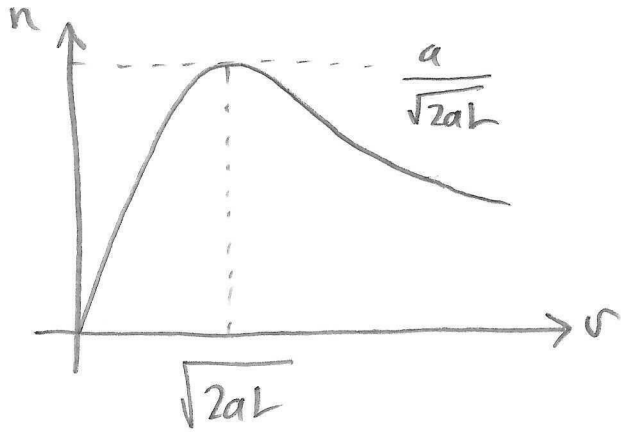
$$\frac{v^2}{a} = \frac{v^2}{2a} + L$$

$$\frac{v^2}{2a} = L$$

$\therefore \boxed{v = \sqrt{2aL}}$

So

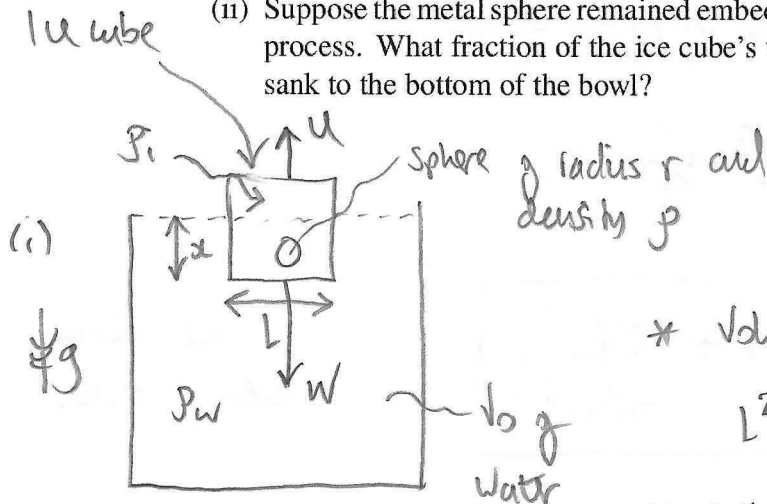
$$n = \frac{v}{\frac{v^2}{2a} + L} \quad \text{and} \quad n_{\text{max}} = \frac{a}{\sqrt{2aL}}$$
$$\text{at } v = \sqrt{2aL}$$



- d) A metal sphere of radius 2.0 cm was embedded in an ice cube with a side length 10 cm floating in a bowl of water. The bowl of water was completely full (i.e., adding any more water would cause it to spill). After the ice cube had completely melted, the water level of the bowl dropped. To make the bowl full again, 60 mL of water had to be added. The densities of water and ice are 1.0 g cm^{-3} and 0.90 g cm^{-3} respectively.

(i) Find the density of the metal.

(ii) Suppose the metal sphere remained embedded in the ice cube throughout the melting process. What fraction of the ice cube's volume had melted when the ice cube just sank to the bottom of the bowl?



* Volume submerged is $L^2 x$.

* Upthrust is weight of water displaced, i.e. $U = \rho_w L^2 x g$

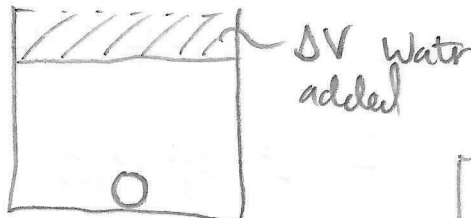
* In e.g., this is balanced by weight of ice cube & sphere.

$$W = \underbrace{\frac{4}{3}\pi r^3 \rho g}_{\text{sphere}} + \underbrace{\left(L^3 - \frac{4}{3}\pi r^3\right) \rho_i g}_{\text{cube with sphere in it.}}$$

Hence if $U = W$:

$$\rho_w L^2 x = \frac{4}{3}\pi r^3 \rho + \left(L^3 - \frac{4}{3}\pi r^3\right) \rho_i \quad (1)$$

After melting:



Conserving volumes of bowl before and after:

$$L^2 x + \cancel{V_0} = \frac{4}{3}\pi r^3 + \Delta V + \cancel{V_0} + \delta V \quad (2)$$

δV is volume of water melted from ice cube.

Conserving mass of H_2O :

$$\rho_i \left(L^3 - \frac{4}{3}\pi r^3\right) = \rho_w \delta V \quad (3)$$

Hence: $L^3 \rho = \frac{4}{3}\pi r^3 + \Delta V + \frac{\rho_i}{\rho_w} (L^3 - \frac{4}{3}\pi r^3)$

and $\therefore$ vsy ①:

$$\begin{aligned} \rho_w \cdot \frac{4}{3}\pi r^3 + \rho_w \Delta V + \rho_i (L^3 - \frac{4}{3}\pi r^3) \\ = \frac{4}{3}\pi r^3 \rho + \rho_i (L^3 - \frac{4}{3}\pi r^3) \end{aligned}$$

$$\Rightarrow \boxed{\rho = \rho_w \left(1 + \frac{\Delta V}{\frac{4}{3}\pi r^3} \right)}$$

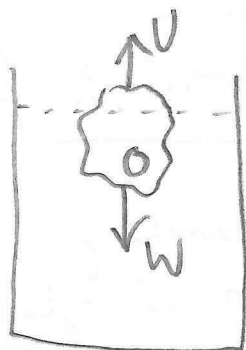
if ice density doesn't matter.

$$\therefore \rho = 1.0 \left(1 + \frac{60}{\frac{4}{3}\pi (2)^3} \right) \approx 1 + \frac{60}{33.51} = \boxed{2.8 \text{ g/cm}^3}$$

(g/cm³)

(ii) Now assume sphere remains embedded in ice cube during melting, and when volume fraction f has melted, ice + sphere starts to sink.

In this, density



$$\boxed{\rho_{\text{remaining ice cube + sphere}} = \rho_w}$$

Mass of ice left is $(1-f)(L^3 - \frac{4}{3}\pi r^3) \rho_i$

$$\therefore \rho_w = \frac{\left\{ (1-f)(L^3 - \frac{4}{3}\pi r^3) \rho_i + \frac{4}{3}\pi r^3 \rho \right\}}{\left\{ (1-f)(L^3 - \frac{4}{3}\pi r^3) + \frac{4}{3}\pi r^3 \right\}}$$

← mass of ice + sphere

$$\therefore (1-f)(L^3 - \frac{4}{3}\pi r^3)(\rho_w - \rho_i)$$

$$= \frac{4}{3}\pi r^3 (\rho - \rho_w)$$

↑ volume of ice + sphere.

$$\therefore 1-f = \frac{\frac{4}{3}\pi r^3 (p-p_w)}{(p_w-p_i)(L^3 - \frac{4}{3}\pi r^3)}$$

$$\therefore f = 1 - \frac{\frac{4}{3}\pi r^3 (p-p_w)}{(p_w-p_i)(L^3 - \frac{4}{3}\pi r^3)}$$

$$\therefore f = 1 - \left(\frac{p-p_w}{p_w-p_i} \right) \left(\frac{L^3}{\frac{4}{3}\pi r^3} - 1 \right)^{-1}$$

$$\text{So } f \approx 1 - \frac{2.8-1.0}{1.0-0.9} \left(\frac{1000}{33.51} - 1 \right)^{-1}$$

$$\approx \boxed{0.38}$$

EMF $\mathcal{E}$

e) A circuit consists of an ideal battery (negligible internal resistance) and three resistors. Two of the resistors labelled R_A have the same resistance and are connected in series with the battery, whilst the third, R_B is connected in parallel with the battery. If R_B and one of the R_A are swapped,

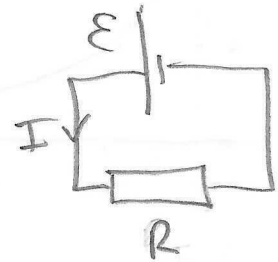
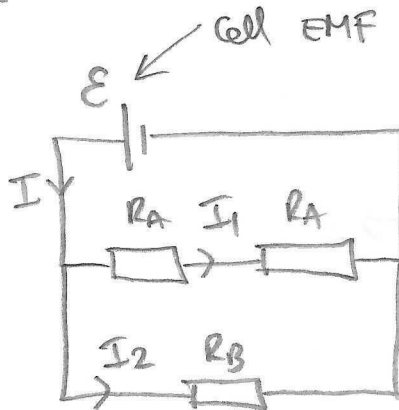
(i) Find the upper limit of the factor by which the power dissipated by the circuit can increase due to this swap.

(ii) If swapping resistors R_A and R_B halves the total dissipated power, what is the ratio

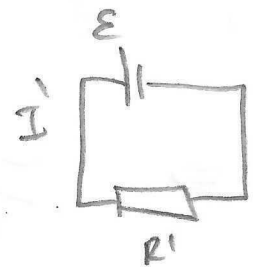
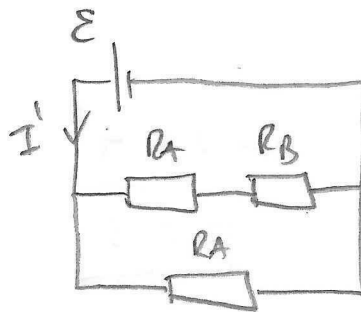
$$\frac{R_A}{R_B} \leftarrow \alpha$$

[6]

Initially :



After Swap:



$$R = \frac{1}{\frac{1}{2R_A} + \frac{1}{R_B}}$$

$$R' = \frac{1}{\frac{1}{R_A + R_B} + \frac{1}{R_A}}$$

$$\therefore R = \frac{2R_A R_B}{R_B + 2R_A}$$

$$\therefore R' = \frac{R_A(R_A + R_B)}{2R_A + R_B}$$

$$P = \frac{\mathcal{E}^2}{R}$$

$$P' = \frac{\mathcal{E}^2}{R'}$$

$$\therefore \frac{P'}{P} = \frac{R}{R'}$$

↑ easier than using $I^2 R$, $I'^2 R'$.

$$\therefore \frac{P'}{P} = \left(\frac{2R_A R_B}{R_B + 2R_A} \right) \left(\frac{2R_A + R_B}{R_A(R_A + R_B)} \right)$$

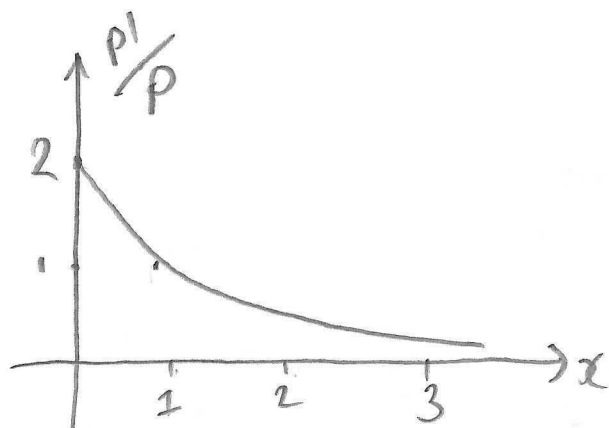
$$\text{Let } \alpha = R_A/R_B \quad \therefore \frac{P'}{P} = \frac{2}{1 + \alpha}$$

$$\therefore \frac{P'}{P} = \left(\frac{2\alpha}{1 + 2\alpha} \right) \left(\frac{1 + \alpha}{\alpha} \right)$$

$$\therefore \boxed{\frac{P^I}{P} = \frac{2}{1+x}}$$

So max value is 2
when $x=0$

ie $R_B \gg R_A$.



when $x \rightarrow 0$,

$$\frac{P^I}{P} \rightarrow 0$$

when $x=1$, $P^I=P$.

$$(ii) \quad \text{If } \frac{P^I}{P} = \frac{1}{2}$$

$$\Rightarrow \frac{1+x}{2} = 2 \Rightarrow 1+x=4$$

$$\Rightarrow \boxed{x=3}$$

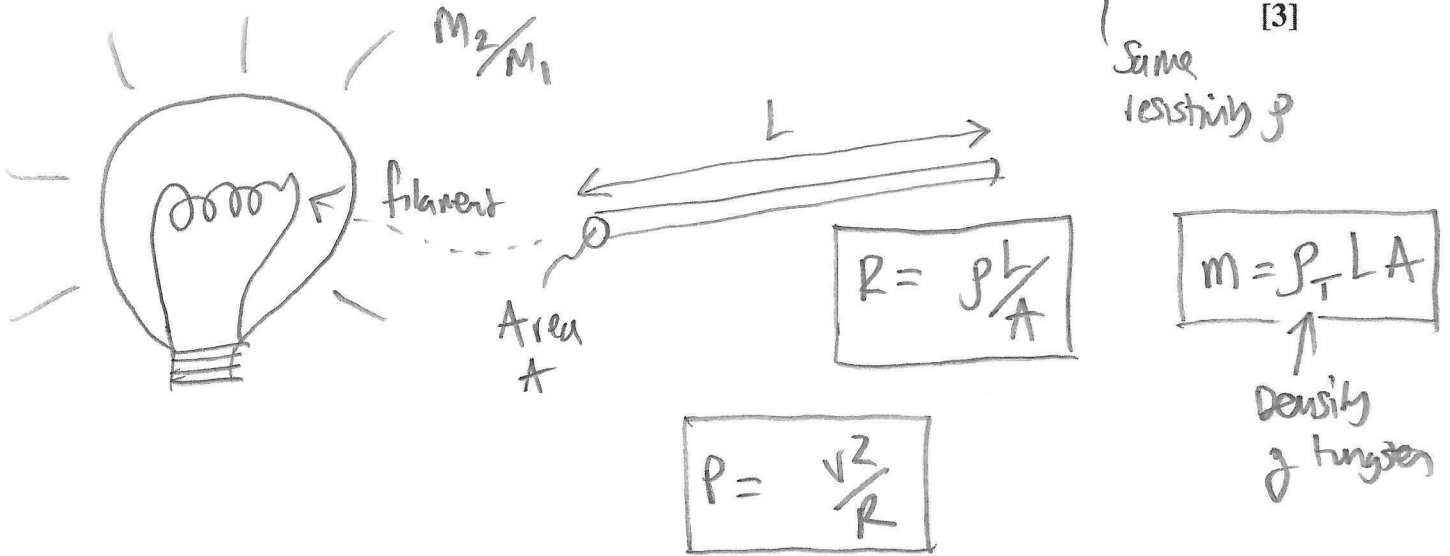
$$\text{So } \frac{R_A}{R_B} = 3 \Rightarrow \boxed{R_A = 3R_B}$$

$$V_2, P_2$$

$$V_1, P_1$$

- f) A household filament bulb has a rating of 240 V, 60 W, whereas a car filament headlamp bulb is rated at 12 V, 60 W. The filaments of the bulbs are both tungsten, of a similar length, and operate at the same temperature in order to produce the same spectrum. Determine the ratio of the masses of the two filaments.

[3]



so if P is the same for both filaments:

$$\frac{V_1^2}{\rho \frac{L}{A_1}} = \frac{V_2^2}{\rho \frac{L}{A_2}} \Rightarrow A_1 V_1^2 = A_2 V_2^2$$

$$\text{so } \frac{A_2}{A_1} = \frac{V_1^2}{V_2^2}$$

$$\text{Now } \frac{M_2}{M_1} = \frac{\rho_T L A_2}{\rho_T L A_1} = \frac{A_2}{A_1} \therefore \frac{M_2}{M_1} = \left(\frac{V_1}{V_2} \right)^2$$

$$\begin{aligned} \therefore \frac{M_2}{M_1} &= \left(\frac{240}{12} \right)^2 \\ &= \left(\frac{2 \times 12 \times 10}{12} \right)^2 \\ &= \boxed{400} \end{aligned}$$

- g) There is a thin rectangle sheet $ABCD$ of uniform resistivity, where $AB = 2BC$, as shown in **Fig. 3**. Two measurements of its resistance using a Ohmmeter have been made. The first is made across the mid-point of AB and the edge CD (a metal band is connected along CD), giving value R_1 . The second is made between A and B , giving value R_2 . Find the ratio $\frac{R_1}{R_2}$.

Show your ideas through diagrams.

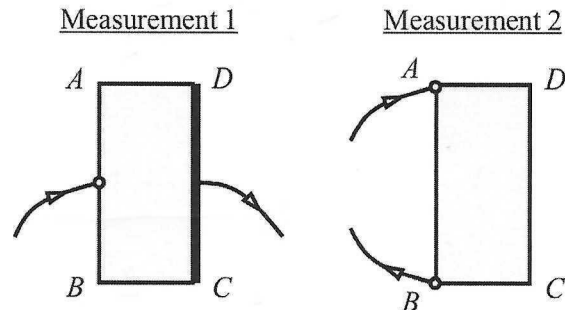
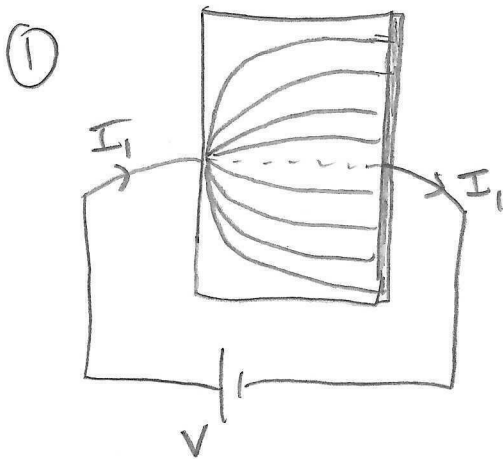


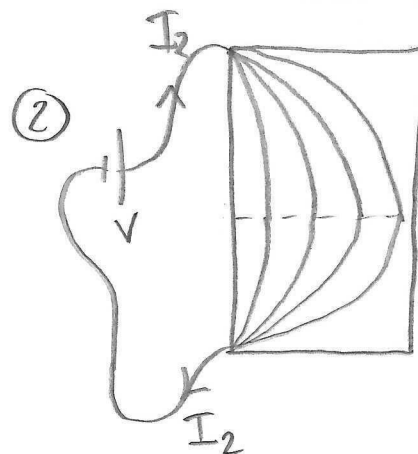
Figure 3: Resistance of a rectangular sheet

[3]

Current flow diagrams are a useful start. Idea is "Can we reduce the problem into a standard resistor series and parallel combination?"

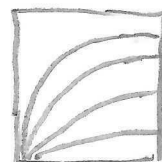


$$R_1 = \frac{V}{I_1}$$



$$R_2 = \frac{V}{I_2}$$

By symmetry, can see that

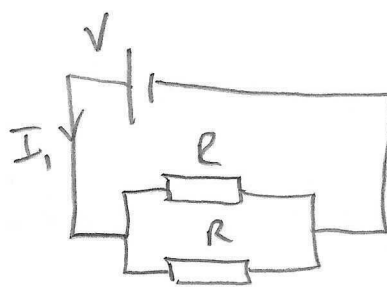


is common to both.

let this have resistance R .

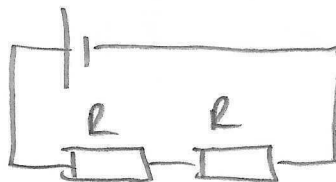
Hence

(1) is



$$\text{So } R_1 = \frac{R}{2}.$$

(2) is



$$\text{So } R_2 = 2R.$$

$$\therefore \frac{R_1}{R_2} = \frac{1}{4}$$

- h) An infinite tree of identical resistors R is constructed as in **Fig. 4**. Three resistors are each connected to pairs of identical resistors which themselves are each connected to pairs stretching out to infinity. If we have a current entering at the centre of the tree of resistors and the entire outer edge is grounded, what is the resistance of the tree?

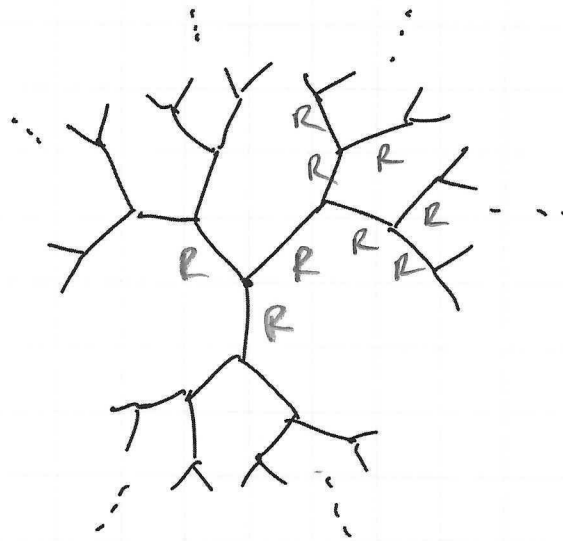
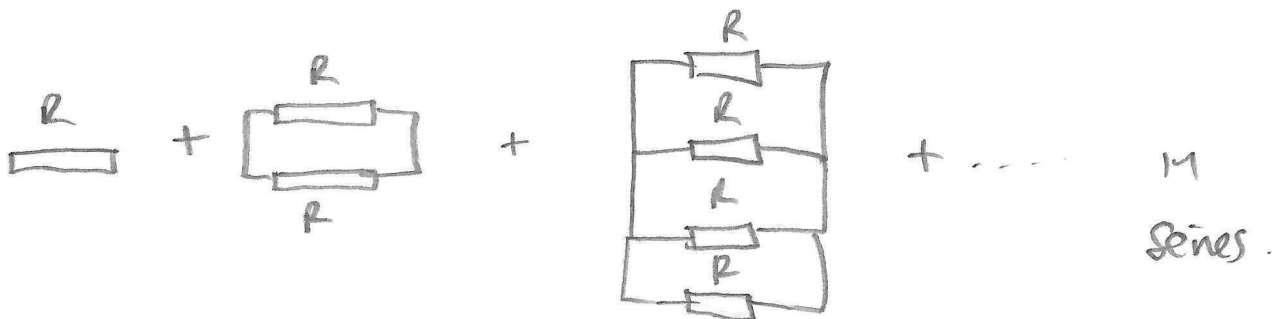


Figure 4: Resistance of tree network of identical resistors.

3 parallel branches of :



$$\text{so } R + \frac{R}{2} + \frac{R}{4} + \dots$$

$$\text{i.e. } R \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right) = 2R.$$

$$\left[a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \right] \quad \text{Geometric progression.}$$

$$\text{if } |r| < 1 \text{ and } a=1, r=\frac{1}{2}$$

$$\Rightarrow \text{Sum is } \frac{1}{1-\frac{1}{2}} = 2$$

$$\text{So a total of } \boxed{\frac{2R}{3}}$$

- i) When we pour hot water into an otherwise cold glass (cup), there is a chance the glass might unfortunately crack. Suppose, as a result of conduction of heat the temperature difference between the inner and outer surface of the glass is ΔT . The following are some information regarding the glass:

Young's modulus $E = 70 \text{ GPa}$

Coefficient of linear expansion $\alpha = 8.0 \times 10^{-6} \text{ K}^{-1}$

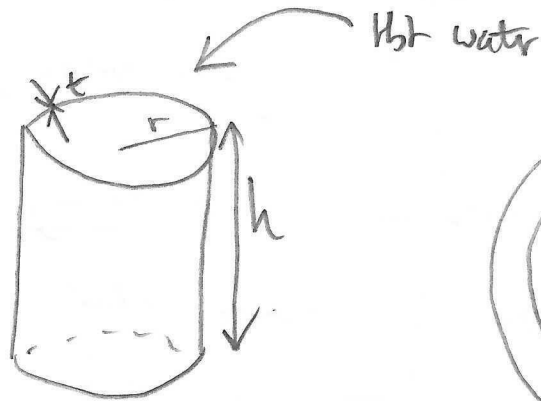
Compressive strength $\sigma = 21 \text{ MPa}$

- (i) By estimating the maximum stress developed in the glass, find the minimum temperature difference ΔT required to make the glass break.

- (ii) Is making the glass wall of the cup thicker a solution to this problem?

[5]

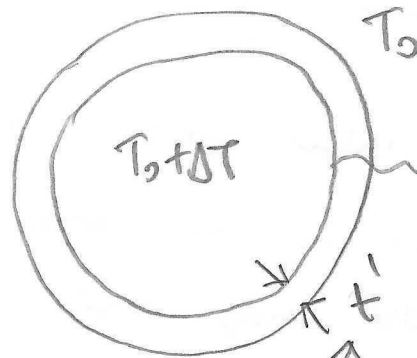
(ii)



glass cup

of radius r (inner radius)

and thickness t .



Inner Circumference

$2\pi r(1 + \alpha \Delta T)$
after water added
and inner wall
heated up to $T_0 + \Delta T$.

May not be
Same?

So inner wall subject
to strain $\epsilon = \frac{2\pi r \alpha \Delta T}{2\pi r}$

$$= \boxed{\alpha \Delta T}$$

'Extreme model':
only inner wall expands?

consider glass
where to find t' from t .

Now stress $\sigma = \epsilon E$ So assume when this
> 21 MPa the glass will break.

$$\therefore \Delta T E > 21 \times 10^6 \text{ Pa}$$

$$\Delta T > \frac{21 \times 10^6}{70 \times 10^9 \times 8.0 \times 10^{-6}} \quad (^\circ\text{C})$$

$$\boxed{\Delta T > 37.5^\circ\text{C}}$$

(ii) * If $t \uparrow$ then conduction slower from inner wall to outer wall.

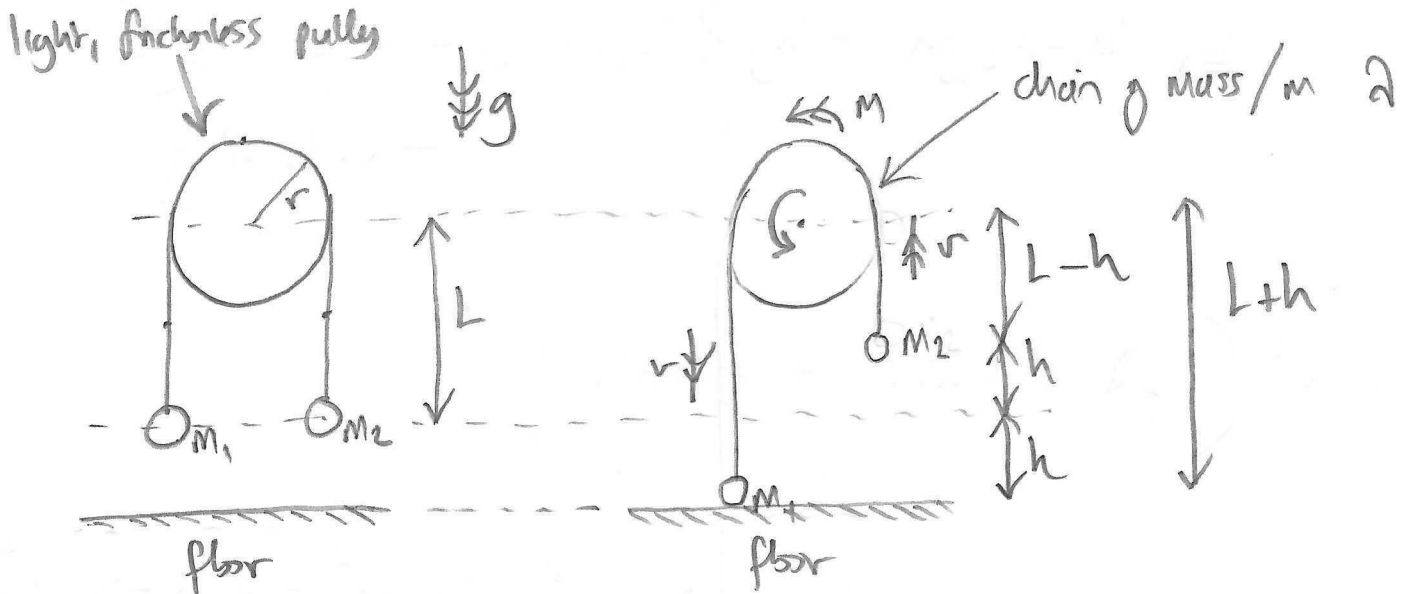
* So ΔT is maximized (for larger) if $t \uparrow$.

* So increasing t will actually improve the chances of $\Delta T > 37.5^\circ\text{C} \Rightarrow$ increasing t will increase the chance of the glass breaking.

This is rather counter-intuitive! A really thin glass cup will/might (!) conduct heat away fast enough that $\Delta T < \frac{\sigma}{\alpha E}$, and $\therefore$ won't break.

- j) Weights m_1 and m_2 hung from a massive chain with a mass of λ per unit length on a light frictionless pulley. The system is released from rest with the weights initial level, and then m_1 reaches the floor ($m_1 > m_2$). By considering the energy of the system, for what ratio of masses $\frac{m_1}{m_2}$ does the value of the mass of the chain not make any difference?

[4]



Initially, total energy is $(m_1 + m_2)gh + 2\lambda Lg(h + \frac{L}{2})$
 1.0 100% GPE $\rightarrow + \pi r \lambda g(L + h + r)$

when m_1 hits the floor, total energy is:

$$m_2 g(2h) + \lambda(L+h)g(\frac{L+h}{2}) + \lambda(L-h)g(2h + \frac{L-h}{2}) + \pi r \lambda g(L+h+r)$$

$$+ \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}\lambda(2L + \pi r)v^2$$

By conservation of energy:

$$(m_1 + m_2)gh + 2\lambda Lg(h + \frac{L}{2}) = 2m_2 gh$$

$$+ \frac{1}{2}\lambda(L+h)^2 g + \frac{1}{2}\lambda(L-h)^2 g + 2\lambda h(L-h)g + \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}\lambda(2L + \pi r)v^2$$

Divide up chain +

$$2 \times \frac{L}{2}$$

MS doesn't consider length of chain round top half of pulley

$$\text{So } v^2 \left(\frac{1}{2}\lambda(2L + \pi r) + \frac{m_1 + m_2}{2} \right)$$

$$= (m_1 - m_2)gh + \lambda g \left\{ 2L(h + \frac{h}{2}) - \frac{(L+h)^2}{2} - \frac{(L-h)^2}{2} - 2h(L-h) \right\}$$

$$\therefore v^2 \left(L\lambda + \frac{1}{2}\lambda\pi r + \frac{1}{2}(m_1 + m_2) \right)$$

$$= (m_1 - m_2)gh + \lambda g \left\{ \cancel{2Lh} + \cancel{L^2} - \cancel{\frac{L^2}{2}} - \cancel{Lh} - \cancel{\frac{h^2}{2}} - \cancel{\frac{L^2}{2}} + \cancel{Lh} - \cancel{\frac{h^2}{2}} - \cancel{2hL} + 2h^2 \right\}$$

$$= (m_1 - m_2)gh + \lambda g \{ h^2 \}$$

$$\text{So } v^2 = \frac{(m_1 - m_2)gh + \lambda h^2 g}{L\lambda + \frac{1}{2}\lambda\pi r + \frac{1}{2}(m_1 + m_2)}$$

MS seems to suggest $L=h$ and $\frac{\pi r}{2} \ll L \dots$

But let's keep generic for now!

$$\text{ie } \boxed{v^2 = hg}$$

$$\therefore v^2 \text{ independent of } \lambda \text{ if } (m_1 - m_2) + 2h = L + \frac{1}{2}(m_1 + m_2) + \frac{1}{2}\pi r$$

$$\frac{m_1}{2} - \frac{3m_2}{2} = \lambda(L - h + \frac{1}{2}\pi r) \quad \text{So if } L=h \text{ and } \frac{\pi r}{2} \ll L$$

$$\Rightarrow L - h + \frac{1}{2}\pi r = 0 \quad \text{and } \therefore \text{equation true } \forall \lambda$$

$$\text{if } \boxed{m_1 = 3m_2}$$

- k) Two tiny conducting spheres A and B of different masses are placed in a smooth hemispherical insulating bowl of radius 10 cm, and connected with an insulating spring with a natural length 8.0 cm and spring constant $k = 120 \text{ N m}^{-2}$, as illustrated in Fig. 5. The charges on A and B are $+2q$ and $+q$ respectively. At equilibrium, the arc length of A to the bottom of the bowl is $d_1 = 7.0 \text{ cm}$, and the arc length of B to the bottom of the bowl is $d_2 = 5.0 \text{ cm}$.

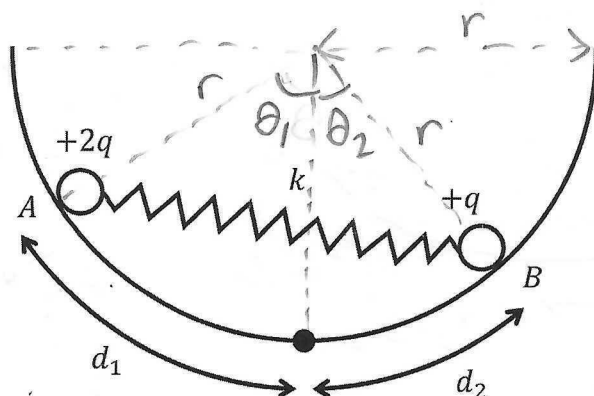
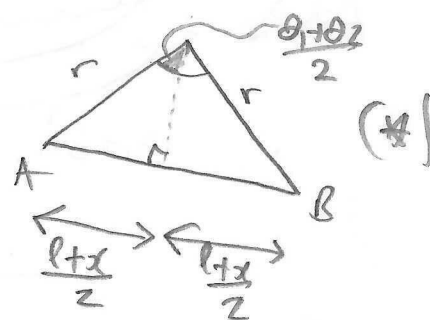


Figure 5: Two charges in a bowl

$$d_2 = r\theta_2$$

$$d_1 = r\theta_1$$



- (i) What is the elastic potential energy stored in the spring?
(ii) Through a discharging process, the charge on A has been halved. This causes A to be displaced by 1.3 cm down along the bowl. How much will B be displaced along the bowl?

[6]

(i) let spring extension be x , $\therefore AB = l + x$
where l is the natural, unextended, length of the spring.

From diagram (*) $r \sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \frac{l+x}{2}$

$$\therefore x = 2r \sin\left(\frac{\theta_1 + \theta_2}{2}\right) - l = \boxed{2r \sin\left(\frac{d_1 + d_2}{2r}\right) - l}$$

EPE is $\boxed{\frac{1}{2} k x^2}$

$$\therefore \text{EPE} = \frac{1}{2} (120) \left(2 \times 0.1 \sin\left(\frac{7+5}{2 \times 10}\right) - 0.08 \right)^2$$

$\underbrace{\hspace{10em}}_{0.0329 \text{ m}}$

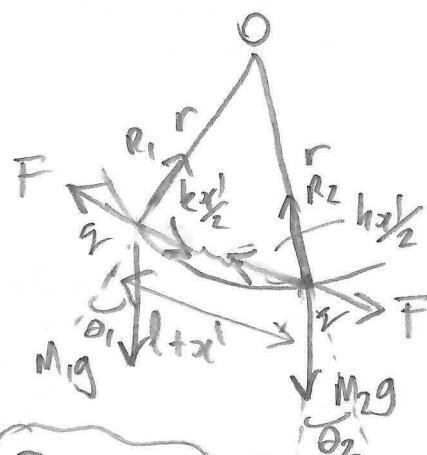
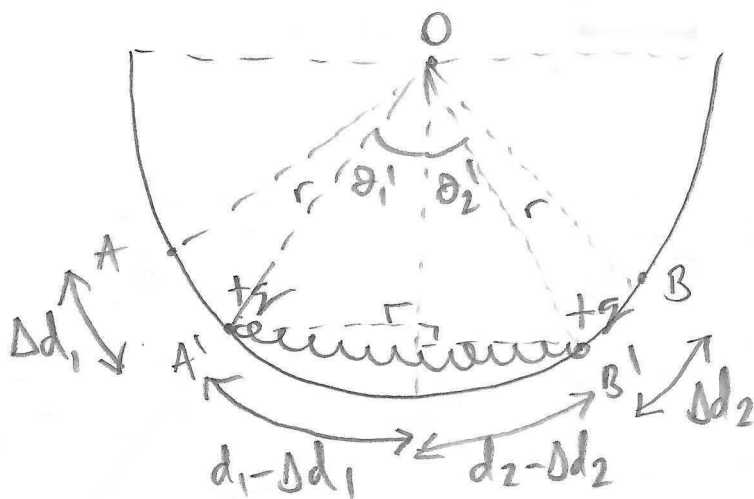
$$= \boxed{0.0655}$$

$$\theta_1 = 0.7 \text{ rad}$$

$$\theta_2 = 0.5 \text{ rad}$$

(iii) Now halve charge on A:

g



So for eq: $m_1 g r \sin \theta_1' = m_2 g r \sin \theta_2'$
to balance moments about O.

By symmetry, the forces of electrostatic repulsion are equal and opposite to the spring restoring force, so neglect any turning moment by these forces about O.

Note no friction (// slope) and no turning moment by normal contact forces R_1, R_2 , since both are radial

So quantity $\frac{\sin \theta_1'}{\sin \theta_2'}$ is conserved, since a similar scenario for the pre-discharge situation.

THIS IS THE KEY TRICK!

ie when considering the whole system these forces cancel.

$$\therefore \frac{\sin \theta_1'}{\sin \theta_2'} = \frac{\sin \theta_1}{\sin \theta_2}$$

We are given $\Delta d_1 = 1.3 \text{ cm}$

and since $r \theta_1' = d_1 - \Delta d_1$

$$\Rightarrow \theta_1' = \frac{d_1 - \Delta d_1}{r} = 0.57 \text{ rad.}$$

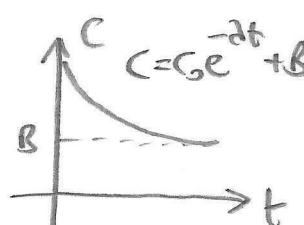
$$\therefore \theta_2' = \sin^{-1} \left(\frac{\sin \theta_1' \sin \theta_2}{\sin \theta_1} \right) = \sin^{-1} \left(\frac{\sin 0.57 \sin 0.5}{\sin 0.7} \right) = 0.413 \text{ rad} \approx 23.7^\circ$$

$$\text{Now } d_2 - \Delta d_2 = r \theta_2'$$

$$\begin{aligned} \therefore \Delta d_2 &= d_2 - r \theta_2' \\ &= 5.0 - 10 \times 0.413 \text{ cm} \\ &= \boxed{0.87 \text{ cm}} \end{aligned}$$

- 1) In an experiment, the count rate of a Geiger counter is the sum of a constant background rate B and that due to the radiation emitted directly from a certain decaying radioactive source. The total count rate was measured over a short time interval at the same time in three consecutive weeks. The rate measurements were C_1 , C_2 and C_3 respectively. Obtain an expression for B in terms of C_1 , C_2 and C_3 .

[4]

Week 1 :	$C_0 + B = C_1$	$\downarrow T$ time interval. $\downarrow T$	
Week 2 :	$C_0 e^{-\lambda T} + B = C_2$		
Week 3 :	$C_0 e^{-2\lambda T} + B = C_3$		

where $T = 1$ week and $\lambda = \frac{\ln 2}{t_{1/2}}$.

and C_0 is the count rate, sans background at week 1

$$\text{so } e^{-\lambda T} = \frac{C_2 - B}{C_0} \quad \left| \quad e^{-2\lambda T} = \frac{C_3 - B}{C_0} \right.$$

$$\therefore e^{-2\lambda T} = \left(\frac{C_2 - B}{C_0} \right)^2$$

$\therefore$ equating expressions for $e^{-2\lambda T}$: $\left(\frac{C_2 - B}{C_0} \right)^2 = \frac{C_3 - B}{C_0}$

$\therefore (C_2 - B)^2 = C_0 (C_3 - B)$. Now $C_0 = C_1 - B$

so $C_2^2 - 2C_2B + B^2 = (C_1 - B)(C_3 - B) = C_1C_3 - BC_3 - BC_1 + B^2$

$\Rightarrow B(C_1 + C_3 - 2C_2) = C_1C_3 - C_2^2$

$$\therefore B = \frac{C_2^2 - C_1C_3}{2C_2 - C_1 - C_3}$$

$\Rightarrow \boxed{C_0 = C_1 - B}$

also $e^{\lambda T} = \frac{C_0}{C_2 - B}$

so $\boxed{\lambda = \frac{1}{T} \ln \left(\frac{C_0}{C_2 - B} \right)}$



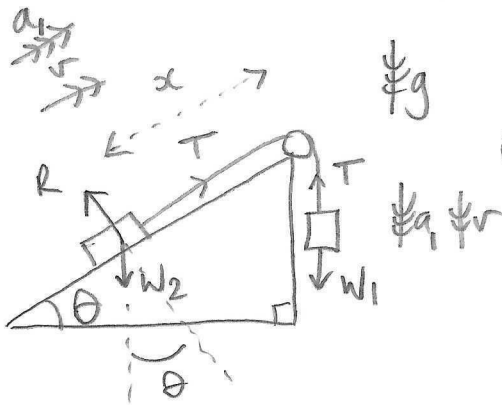
masses

$$M_1 = w_1/g$$

$$M_2 = w_2/g$$

- m) Two weights, w_1 and w_2 are connected by a string which passes over a smooth pulley at the top of a smooth, inclined plane at 30° to the horizontal. One weight rests on the plane whilst the other hangs freely. They can be interchanged to compare their weight. w_1 draws w_2 up the plane in half the time that it takes for w_2 to draw up w_1 . What is the relation between the weights w_1 and w_2 ?

[5]



$$\text{NII: } M_1 a_1 = w_1 - T$$

$$M_2 a_1 = T - w_2 \sin \theta$$

$$0 = R - w_2 \cos \theta$$

// slope

⊥ slope

$$\text{so } (M_1 + M_2) a_1 = w_1 - w_2 \sin \theta$$

$$a_1 = \frac{w_1 - w_2 \sin \theta}{w_1 + w_2} g$$

$$\text{let } x = \frac{1}{2} a_1 t_1^2$$

$$\text{Now if } w_1 \leftrightarrow w_2,$$

$$t_2 = 2t_1$$

$$x = \frac{1}{2} a_2 (2t_1)^2$$

$$\text{where } a_2 = \frac{w_2 - w_1 \sin \theta}{w_1 + w_2} g$$

So by equating x

$$\frac{\frac{1}{2} (w_2 - w_1 \sin \theta) g (2t_1)^2}{w_1 + w_2} = \frac{\frac{1}{2} (w_1 - w_2 \sin \theta) g t_1^2}{w_1 + w_2}$$

$$\frac{1}{2} a_2 t_2^2 = \frac{1}{2} a_1 t_1^2$$

$$\Rightarrow 4w_2 - 4w_1 \sin \theta = w_1 - w_2 \sin \theta$$

$$w_2 (4 + \sin \theta) = w_1 (1 + 4 \sin \theta)$$

$$\therefore \frac{w_1}{w_2} = \frac{4 + \sin \theta}{1 + 4 \sin \theta}$$

$$\text{if } \theta = 30^\circ \Rightarrow \sin \theta = \frac{1}{2}$$

$$\therefore \frac{w_1}{w_2} = \frac{9}{3}$$

$$\therefore \frac{w_1}{w_2} = \boxed{\frac{3}{2}}$$

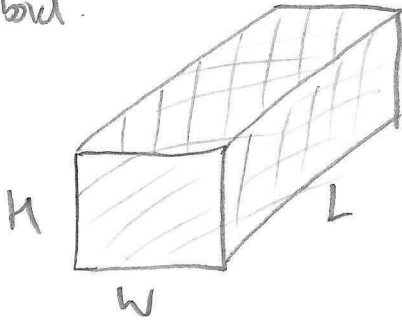
- n) Radiation is attenuated when it passes through material. The intensity decreases exponentially with distance from the surface of the material. A cuboid is made from a material which halves the radiation at a depth of 1.0 cm below the surface. When a beam of radiation is directed in turn along the three axes of the cuboid, the intensities are reduced to 60%, 70%, 80% of the incident. Determine the volume of the cuboid.

I_W, I_H, I_L

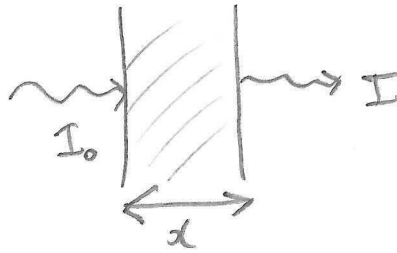
✓

[4]

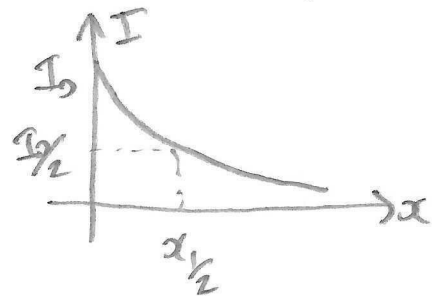
cuboid.



$$V = WHL$$



$$I = \frac{I_0}{2^{x/x_{1/2}}}$$



$$S_o \quad I_W = \frac{I_0}{2^W}$$

$$I_L = \frac{I_0}{2^L}$$

$$I_H = \frac{I_0}{2^H}$$

} W, H, L in cm
so we can omit $x_{1/2}$
Since $x_{1/2} = 1 \text{ cm}$.

$$\therefore W = \frac{\ln I_0/I_W}{\ln 2}, \quad L = \frac{\ln I_0/I_L}{\ln 2}, \quad H = \frac{\ln I_0/I_H}{\ln 2}$$

$$\therefore \text{Volume } V = \frac{1}{(\ln 2)^3} \ln\left(\frac{1}{0.6}\right) \ln\left(\frac{1}{0.7}\right) \ln\left(\frac{1}{0.8}\right)$$

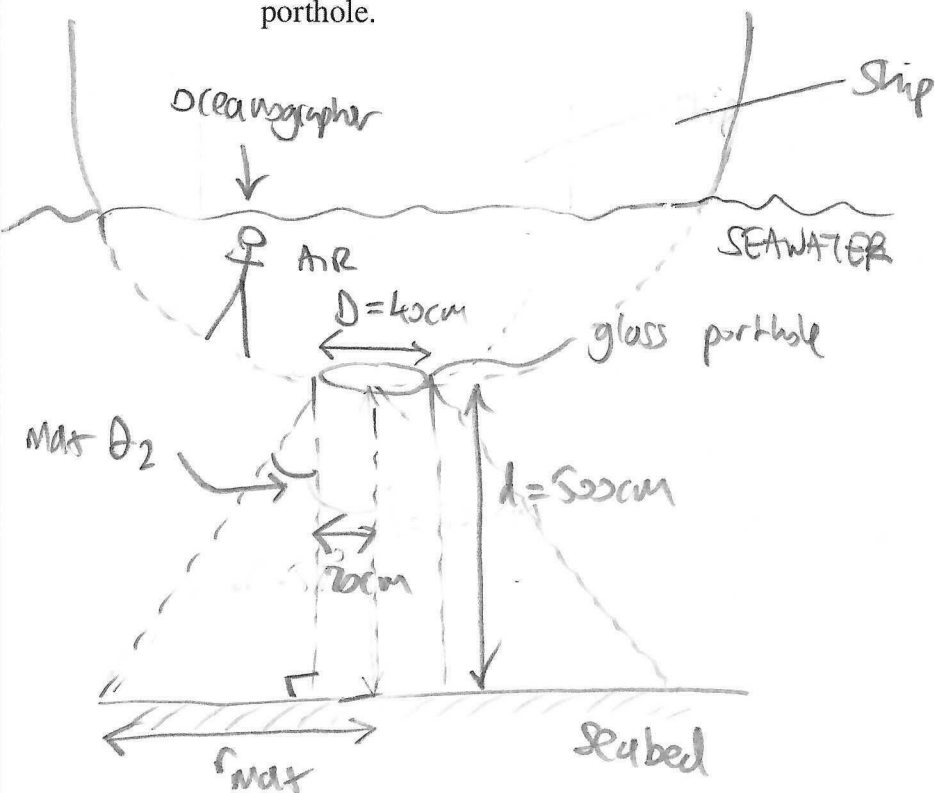
$$= \boxed{0.122 \text{ cm}^3}$$



It would help if stated that the porthole was directly underneath the hull!

- o) An exploration ship has a circular glass filled circular porthole in the hull of the ship for studying the seabed. The porthole has a diameter of 40 cm which is much greater than the thickness of the glass. Calculate the area of seabed that is visible through the porthole if the refractive index of seawater is $n_w = 1.4$ and the seabed is 5.0 m below the porthole.

[4]



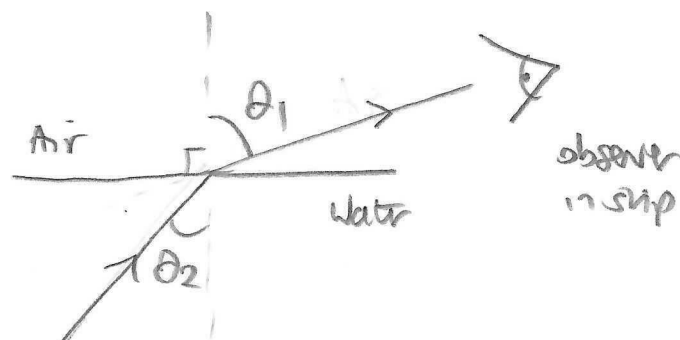
So ignore refraction in the glass. Treat the problem as refraction between air (inside ship) $n_1 = 1.00$ and water $n_2 = 1.4$.

Snell's law:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

largest angle θ_2 to reach observer in ship is when $\theta_1 = 90^\circ$ if θ_2 is the critical angle.

$$\therefore \text{max } \theta_2 = \sin^{-1} \left(\frac{1.00 \sin 90^\circ}{1.4} \right) = 45.58^\circ$$



AND CONSIDER EDGES OF THE PORTHOLE

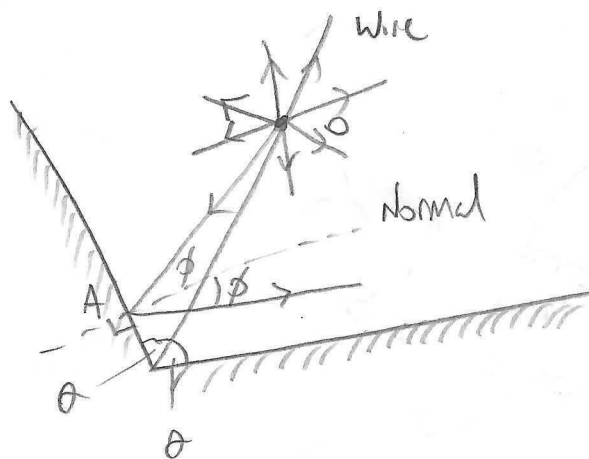
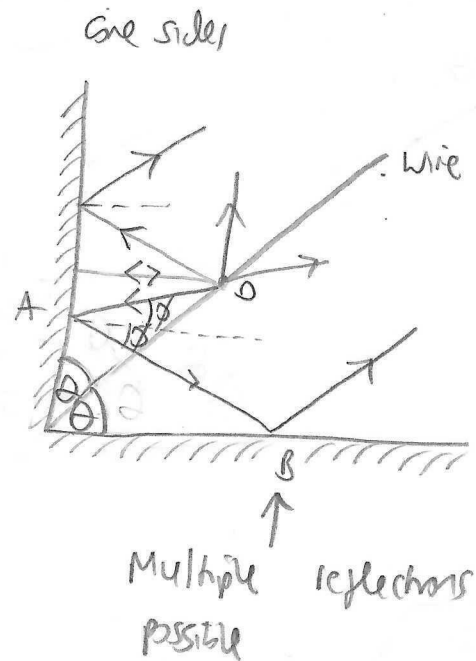
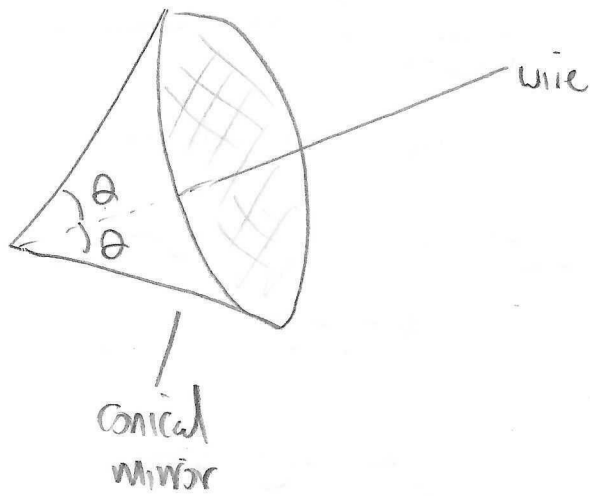
$$\text{So } r_{\text{max}} = 500 \text{ cm } \tan 45.58^\circ + 20 \text{ cm} = 530.3 \text{ cm} = 5.30 \text{ m}$$



$$\text{Area of seabed is } \pi r_{\text{max}}^2 = 88.3 \text{ m}^2$$

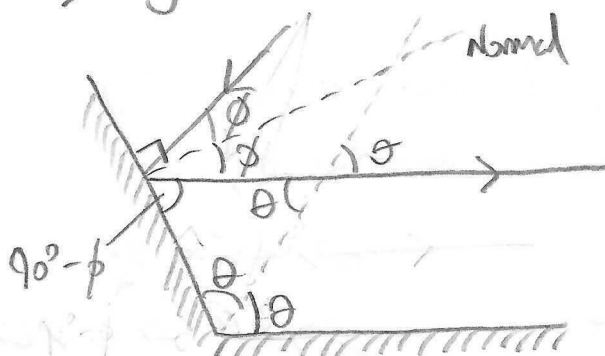
- p) The inner surface of a right circular cone is coated with a mirror-like reflecting material to form a conical mirror. A thin straight wire heated to white heat is stretched along the inner axis of the conical mirror. What is the minimum angle θ of the cone such that rays of light emitted by the filament will not be reflected off the cone more than once.

[4]



Wan't limits
on θ and ϕ

Consider ray source at O on the wire. It strikes one "edge" of the cone at A. If reflected ray is $\parallel$ to lower edge then this is the limiting case of no further reflection. i.e. greater $\theta \Rightarrow$ no further reflection possible.



$$\text{So } 180^\circ = 90^\circ - \phi + 2\theta$$

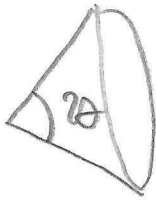
$$\Rightarrow 90^\circ = 2\theta - \phi$$

so for no extra reflection

$$\theta > \frac{90^\circ + \phi}{2} \quad \forall \phi$$

So Now $\theta > \frac{90^\circ + \phi}{2}$ implies $\theta > 90^\circ$ if $\phi_{\max} = 90^\circ$

$\Rightarrow$ which invalidates the assumption of a single reflection $\theta < 180^\circ$.



what range of ϕ are allowed

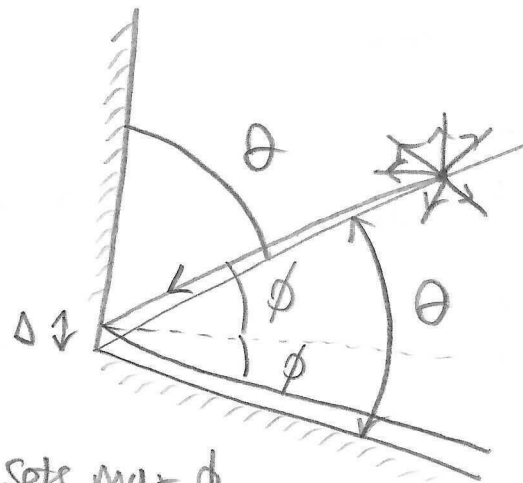
In this situation

or could imagine

$$\theta = 2\phi \quad \text{But}$$

The ray strikes 'just' above the gap by Δ and the angle $\theta = 2\phi$ results in no further reflections.

An angle of incidence ϕ larger than when $2\phi = \theta$ will result in multiple reflections



sets max ϕ

This is $\approx$ the diagram in the MS.

So

if

$$\theta = 2\phi \quad \text{and}$$

$$90^\circ = 2\theta - \phi$$

sets min θ for no reflection extra

then

$$90^\circ = 2\theta - \theta/2$$

$$90^\circ = \frac{3\theta}{2}$$

$$\boxed{60^\circ = \theta}$$

is the minimum θ for no extra reflections.

$$\left[\text{Strictly: for no extra reflections} \right. \quad \left. \begin{array}{l} \phi < \theta/2, \quad \theta > 45^\circ + \frac{\phi}{2} \\ \therefore \theta > 45^\circ + \frac{\theta}{4} \\ \frac{3\theta}{4} > 45^\circ \Rightarrow \theta > 60^\circ \end{array} \right]$$

Not very satisfying.... But

- q) The tractive force of a locomotive is plotted vertically with the speed horizontally in the graph of **Fig. 6**. Also shown on the same axes is the resistance of the train against its speed. If the weight of the train is 336 tonnes determine approximately the distance travelled before the train reaches its maximum speed.

1 tonne = 1000 kg

idea could be used as a plot.

Tractive and Resistive forces acting on a train

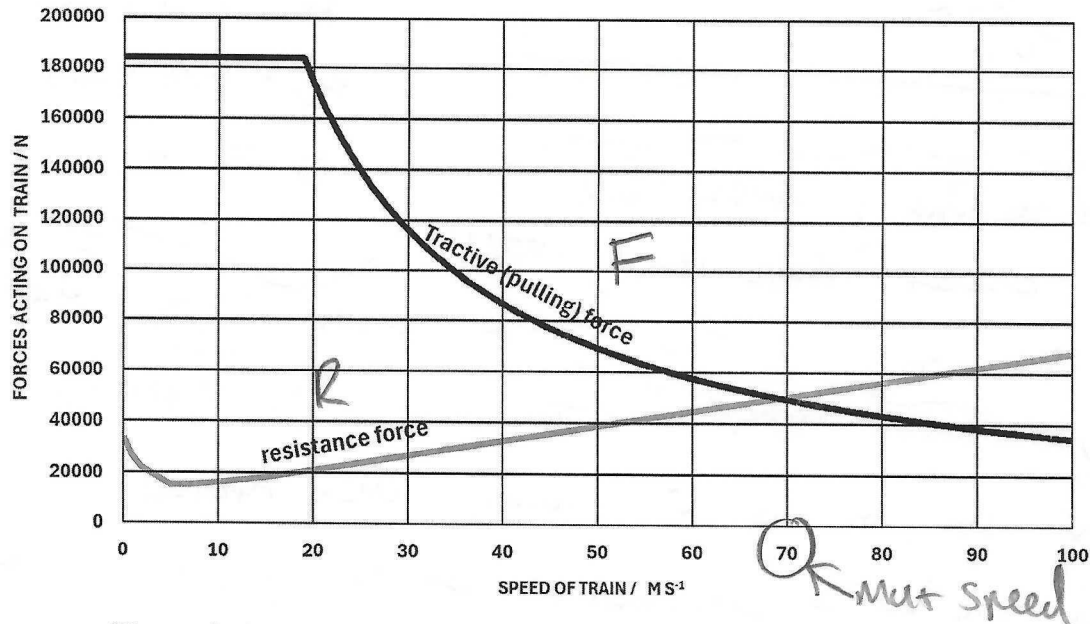
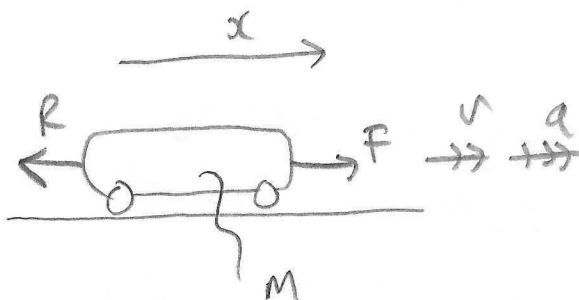


Figure 6: Graph showing the tractive and resistive forces acting on a train.



[4]

NI: $M a = F - R$

so when $F = R$ at $v = 70 \text{ m/s}$ the forces balance and $a \rightarrow 0$.

$\therefore$ since $F > R$ until $v = 70 \text{ m/s}$ this means $v_{\text{max}} = 70 \text{ m/s}$

i.e. if $v > 70 \text{ m/s}$ then $R > F$ and it would decelerate ($a < 0$), reduce v until it gets back to 70 m/s .

could use

$$\Delta \left(\frac{1}{2} M v^2 \right) = \underbrace{(F - R) \Delta x}_{\text{work done}}$$

change in KE

$\rightarrow$ which comes directly from $a = v \frac{dv}{dx}$

$$\therefore M v \frac{dv}{dx} = F - R$$

$$\left[\frac{1}{2} M v^2 \right]_{v_1}^{v_2} = \int_{x_1}^{x_2} (F - R) dx$$

But since F and R are specified as functions of v :

$$\int_0^{70} \frac{v}{F-R} dv = \frac{\alpha}{m}$$

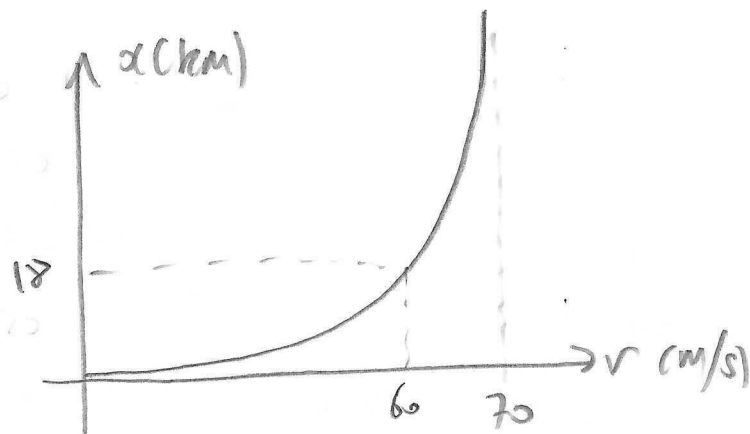
← where α is the distance travelled before max speed reached by the train.

MS uses the idea of constant force (resultant) between 10m/s velocity intervals, but you can clearly see that $F-R$ varies quite significantly.

← i.e. tabulate $\Delta KE, F-R, \Delta \alpha$.

Perhaps a better idea is to approximate the F vs v and R vs v curves, then integrate.

Could do this via a spreadsheet or computer program ... but therefore this doesn't make for a particularly satisfying BPhO question!



Using a MATLAB program (interpolation of $\approx$ samples from graphs)

I make 18 km at 60 m/s

$\approx$ 30 km at 65 m/s

→ and of course an asymptote $\uparrow$ as $v \rightarrow 70$ m/s.

Probably 80 km is a reasonable approx at 70 m/s?

Potentially ☺

REQUIRES AN IMAGINATIVE LEAP!

- r) As shown in Fig. 7, the ends of a massless, rigid rod of length L are attached to a set of rails which form a right angle to each other. The endpoints of the rod can slide freely without friction on the rails. A mass m is attached to the mid-point of the rod.

Find the period T of small oscillations of the rod about its equilibrium position, and express your answers in terms of L , m and g .

[3]

There is a nice trick here — the mass at the centre of the rod moves in a circle of radius $\frac{L}{2}$

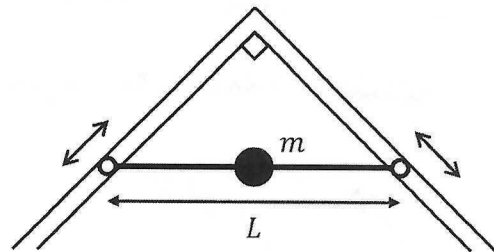


Figure 7: A massless, rigid rod attached to frictionless rails



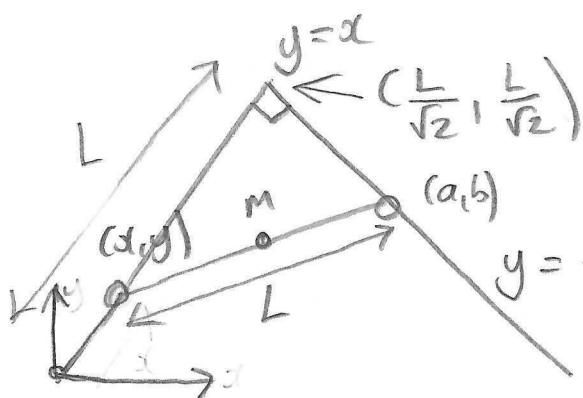
So mass behaves like a pendulum bob of length $l = \frac{L}{2}$

For a pendulum: $T = 2\pi\sqrt{\frac{l}{g}}$

$$T = 2\pi\sqrt{\frac{L}{2g}}$$

This is not the best way to see the trick!

Constrain rod to swaddle right angle



$$y = -x + C \leftarrow \frac{L}{\sqrt{2}} = -\frac{L}{\sqrt{2}} + C \Rightarrow C = L\sqrt{2}$$

$\uparrow$
 $\frac{2}{\sqrt{2}}L$

$$L^2 = (a-x)^2 + (b-y)^2$$

$$y = x$$

$$\therefore L^2 = (a-x)^2 + (b-x)^2$$

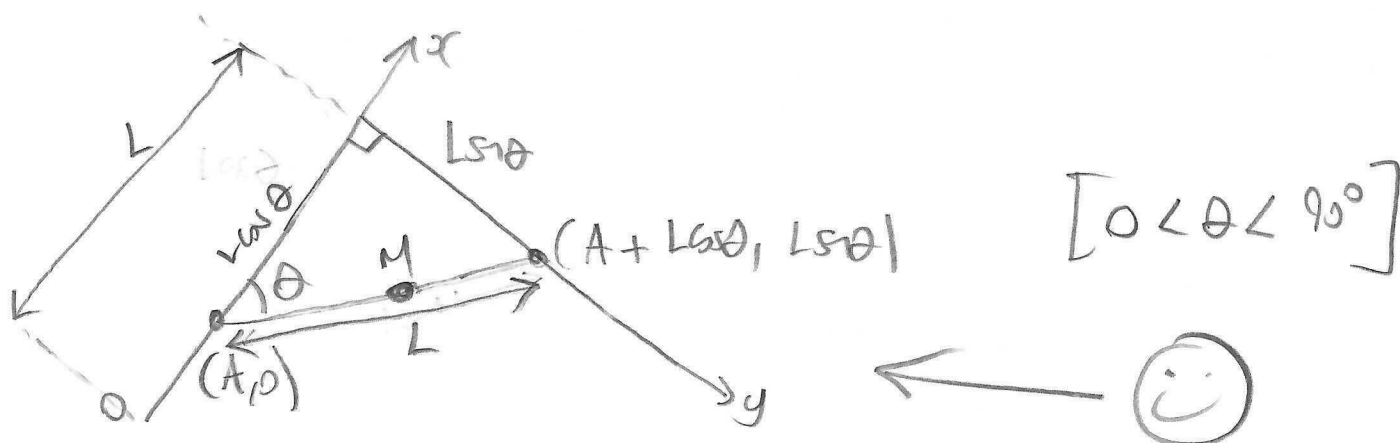
$$b = -a + L\sqrt{2}$$

$$\begin{aligned} \text{so } L^2 &= (a-x)^2 + (L\sqrt{2} - a - x)^2 \\ L^2 &= a^2 - 2ax + x^2 + (L\sqrt{2} - x)^2 - 2(L\sqrt{2} - x)a + a^2 \\ &= 2L\sqrt{2}a - 2a^2 - 2ax + 2x^2 \end{aligned}$$

$$\therefore 2a^2 - a(2x + 2L\sqrt{2} - 2x) + x^2 + (L\sqrt{2} - x)^2 - L^2 = 0$$

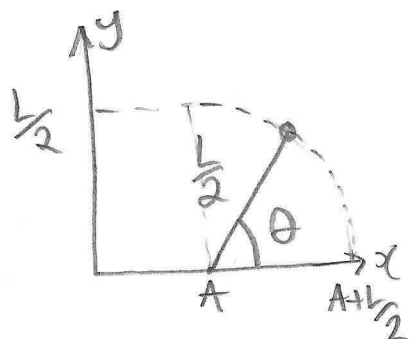
Now M is a COM of rod $\Rightarrow \left(\frac{x+a}{2}, \frac{y+b}{2} \right)$

... This looks algebraically unwieldy ... So choose a better coordinate system that reflects the symmetry of the problem!



$$\vec{OM} \text{ is } \left(\frac{A + A + L \cos \theta}{2}, \frac{L \sin \theta}{2} \right)$$

$$= \begin{pmatrix} A \\ 0 \end{pmatrix} + \frac{L}{2} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad \text{is a circular arc.}$$



Assume small oscillations about $\theta = 45^\circ$.

